THE TURBULENT BOUNDARY LAYER OF DISSOCIATED GAS IN THE INITIAL SECTION OF A TUBE

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ABSTRACT: Many theoretical and experimental papers [1-4] have been devoted to investigating the turbulent boundary layer in the initial section of a channel. For the most part, however, the flow of an incompressible fluid with constant parameters is considered. There are many practical cases in which it is of interest to treat the development of the turbulent boundary layer of gas in the initial section of a pipe when conditions are strongly nonisothermal. A solution of a problem of this type, based on the theory of limit laws, is given in paper [1]. The present article extends this solution to the case of the flow of a high-enthalpy gas when the effect of gas dissociation on the turbulent boundary layer characteristics must be taken into account. We shall consider the flow of a mixture of i gases which is in a "frozen" state inside the boundary layer, and in an equilibrium state on its boundaries. Formulas are derived for the laws of friction and heat exchange, and a solution is given for the turbulent boundary layer equations in the initial section of the pipe when the wall temperature is constant and the gas flows at a subsonic velocity.

§ 1. The relative law of friction for a turbulent boundary layer of dissociated gas. According to paper [1], the relative coefficient of turbulent friction is determined by the expression

$$\Psi=\left[\frac{1}{Z}\int_{\omega_1}^{1}\left(\frac{\rho}{\rho_0}\frac{\tau_0^*}{\tau^*}\right)^{1/2}d\omega\right]^2,\quad Z=\left(\frac{c_{f_0}}{2}\right)^{1/2}\int_{\xi_1}^{1}\left(\frac{\tau_0^*}{1-\beta}\right)^{1/2}\frac{d\xi}{l}$$

$$\Psi=\left(\frac{c_f}{c_{f_0}}\right)_{R^{++}},\qquad c_f=\frac{2\tau_w}{\rho_0 w_0^2},\qquad c_{f_0}=\frac{2\tau_{w0}}{\rho_0 w_0^2}. \tag{1.1}$$

Here Ψ is the ratio of the friction coefficient in the present circumstances to that under standard conditions for the same values of R^{++}.

The relation between the gas density and enthalpy may be obtained from the equations

$$p=\rho RT,\qquad h_0=\sum_i \alpha_{i0}h_{i0},\qquad h_i=\int_0^T c_{p_i}dT+h_i^\circ. \tag{1.2}$$

The degree of dissociation of air for a low-temperature plasma (T = 5000° K) is of the order 0.3. In this region the parameter μc_p may be taken to be constant over the boundary layer cross section. Then

$$\frac{\rho}{\rho_0}=\frac{h_0-\Sigma\alpha_{i0}h_i^\circ}{h-\Sigma\alpha_i h_i^\circ}. \tag{1.3}$$

Here and in what follows the summation is carried out over the index i. Setting (1.2) into (1.3) and assuming similarity of the velocity, enthalpy, and concentration fields, [2] we obtain

$$\frac{\rho}{\rho_0}=\frac{1-\Sigma\alpha_{i0}\psi_i}{\psi_h-\Sigma\alpha_{wi}\psi_i+\{1-\psi_h-\Sigma(\alpha_0-\alpha_w)_i\psi_i\}\omega}$$
$$\left(\psi_h=\frac{h_w}{h_0},\ \psi_i=\frac{h_i^\circ}{h_0}\right). \tag{1.4}$$

The following symbols have been used in Eqs. (1.1)-(1.4): p is the pressure, ρ is the density, h is the enthalpy, h_i° is the heat of formation for the i-th component, d is the degree of dissociation, l is the relative mixing length, ω is the dimensionless velocity, T is the temperature, R is the gas constant; the subscripts are w (parameters at the wall), 0 (parameters in the core of the stream).

Equation (1.4) establishes a relation between the density and velocity of the gas at a given point for given gas parameters at the boundary layer borders. Strictly speaking, Eq. (1.4) is exact only in the case in which the gas flow is without gradient, for a constant value of enthalpy at the wall, and a Prandtl number $P_T = 1.0$.

However, calculations show that this equation may also be used, accurately enough for all practical purposes, where the boundary conditions are more complicated, in particular, for gas flow in the initial section of a pipe.

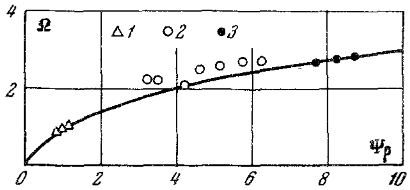

Fig. 1. The effect of nonisothermal conditions on the parameters of the viscous sublayer. The curve is calculated from formula (1.7), the points 1, 2, 3 correspond to the experiments of papers [4-6].

In order to estimate the parameter Z we shall make the usual assumptions of semiempirical turbulence theory: $\beta=0$, $l=\varkappa\xi$ and $\tau_0^*=1.0$. Then

$$Z=-\sqrt{1/2c_{f_0}}\,\varkappa^{-1}\ln\xi_1. \tag{1.5}$$

Analysis of experimental data concerning the velocity distribution over the cross section of a turbulent boundary layer for markedly nonisothermal conditions [4, 6] reveals that the dimensionless velocity at the boundary of the viscous sublayer, and the relative thickness of the viscous sublayer retain the same values as for an incompressible fluid if the physical parameters of the gas are determined from the wall temperature.

Consequently

$$\frac{w_1}{w_0\sqrt{1/2c_f\psi_\rho}}=\frac{\delta_1 w_0\sqrt{1/2c_f\psi_\rho}}{\nu_w}=11.6, \tag{1.6}$$

from which

$$\omega_1=11.6\sqrt{1/2c_{f_0}\Psi\psi_\rho},\qquad \xi_1=\frac{135\psi_\rho\delta^{++}}{\omega_1 R^{++}\delta}, \tag{1.7}$$

$$R^{++}=\frac{\rho_0 w_0\delta^{++}}{\mu_w},\qquad \psi_\rho=\frac{\rho_0}{\rho_w}. \tag{1.8}$$

Formula (1.7) is compared with the experiments of different authors [4, 6] in Fig. 1, where $\Omega=w_1/11.6\cdot w_0\sqrt{1/2c_f}$. Substituting (1.7) into (1.5) we have

$$Z=-\frac{\sqrt{1/2c_{f0}}}{\varkappa}\ln\frac{135\psi_\rho\delta^{++}}{\omega_1 R^{++}\delta}. \tag{1.9}$$

The friction coefficient c_{f_0} for standard conditions is determined from the Karman formula

$$1/2c_{f0}=(2.5\ln R^{++}+3.8)^{-2}. \tag{1.10}$$

It has been shown in papers [7, 8] that the Reynolds number appearing in formula (1.10) should be determined from (1.8).

For standard conditions we have [1]

$$Z=1-\omega_{10}. \tag{1.11}$$

This simple relation between Z and ω_1 may be expected to hold even for more complicated conditions. Actually when calculations made from formula (1.9) (Fig. 2) are compared with formula (1.11),

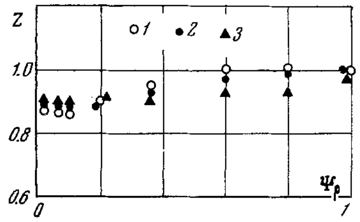

Fig. 2. The effect of nonisothermal conditions on the parameter $Z = z/(1 - \omega_1)$. Formulas (1.9) and (1.7) were used for calculating Z and ω_1, respectively; the different points signify: 1) $R^{++} = 500$, 2) $R^{++} = 10^4$, 3) $R^{++} = 10^8$.

employing the "one seventh" law to determine the velocity distribution, and the limiting function ψ and (1.3) for the various values of ψ_ρ and R^{++}, we see that in regions of variation of ψ_ρ and R^{++} of practical interest we may use the formula

$$Z = 1 - 11.6\sqrt{{}^1/{}_2 c_{f0}\Psi\psi_\rho}\,. \qquad (1.12)$$

Setting Eqs. (1.4), (1.7), and (1.12) into Eq. (1.1) and assuming that (1.9) will be valid when there is dissociation, we may integrate to obtain the relative law of friction for a turbulent boundary layer for the conditions under consideration:

$$\psi_h = \Big[(1 - \Sigma\alpha_{i0}\psi_i)(2 - \sqrt{\Psi})^2 + \Psi\Sigma\alpha_{wi}\psi_i - $$
$$- [1 - \Sigma(\alpha_0 - \alpha_w)_i\,\psi_i]\, 8.2\sqrt{c_{f0}\psi_\rho\Psi}\Psi\Big] \times$$
$$\times \Big[\Psi(1 - 8.2\sqrt{c_{f0}\psi_\rho\Psi})\Big]^{-1}. \qquad (1.13)$$

If $\alpha_0 = \alpha_W = 0$, then

$$\psi_h = 1 + \frac{4(1 - \sqrt{\Psi})}{\Psi(1 - 8.2\sqrt{c_{f0}\Psi\psi_h})}\,. \qquad (1.14)$$

As $R^{++} \to \infty$, $Z \to 1$, $\omega_1 \to 0$ Eq. (1.13) gives

$$\Psi_\infty = 4\left[1 + \left(\frac{\psi_h - \Sigma\alpha_{wi}\psi_i}{1 - \Sigma\alpha_{0i}\psi_i}\right)^{1/2}\right]^{-2}. \qquad (1.15)$$

For a nondissociated gas we obtain the familiar result [1]

$$\Psi_\infty = 4(\sqrt{\psi_h} + 1)^{-2}\,. \qquad (1.16)$$

The maximum discrepancy between formulas (1.14) and (1.16) for $R^{++} = 300$ $\psi_h = 0.1$ is about 25%. Thus for practical calculations we may use the limiting relative law of friction in the form (1.15).

Figure 3 compares the results of calculations from formula (1.13) with the experiments of N. M. Belyanin [3]. We note that the proposed theory agrees satisfactorily with experiment.

§ 2. **The Development of a Dynamic Turbulent Boundary Layer in the Initial Section of a Pipe.** We shall represent the integral momentum equation [1] for the entrance section of a pipe in the form

$$\frac{dR^{++}}{dX} + \frac{R^{++}}{W_0}\frac{dW_0}{dX}(1 + H) = R_{D1}\Psi\frac{c_{f0}}{2}\,. \qquad (2.1)$$

Here $W_0 = w_0/w_{01}$ is the relative velocity in the undisturbed flow core; W_{01} is the velocity at the inlet of the pipe.

Equation (2.1) contains the parameter H, which is the ratio of the displacement thickness to the momentum thickness. We shall examine the effect which nonisothermal conditions and dissociation have on the size of this parameter. The physical displacement thickness δ^+ is related to the displacement thickness in Dorodnitsin variables δ_1^+ by the relation

$$\delta^+ = \delta_1^+ - \int_0^\delta \left(\frac{\rho}{\rho_0} - 1\right)\left(1 - \frac{y}{R_0}\right)dy\,. \qquad (2.2)$$

Taking Eqs. (1.4) into account we have

$$\delta^+ = \delta_1^+ - \int_0^\delta \frac{(h - \Sigma\alpha_i h_i^\circ) - (h_0 - \Sigma\alpha_{i0}h_i^\circ)}{h_0 - \Sigma\alpha_{i0}h_i^\circ}\left(1 - \frac{y}{R_0}\right)dy\,, \qquad (2.3)$$

$$\delta^+ = \delta_1^+\left\{1 - \frac{1}{1 - \Sigma\alpha_{i0}\psi_i}[1 - \psi_h - \sum_i(\alpha_0 - \alpha_w)_i\,\psi_i]\right\}\,. \qquad (2.4)$$

These lead to

$$H = H_1\psi_\alpha, \qquad \psi_\alpha = \frac{\psi_h - \Sigma\alpha_{wi}\psi_i}{1 - \Sigma\alpha_{0i}\psi_i}\,. \qquad (2.5)$$

The form parameter H_1 depends only feebly on the fact that conditions are nonisothermal and may be regarded as a constant equal to $H_{10} = 1.347$.

Figure 4 shows the results of calculations of the parameters of an axially symmetric turbulent boundary layer assuming a wall velocity profile (n = 1/7) and using formula (1.4) for the density.

It is clear from the graph that the parameters δ_1^+ and δ^{++} have a marked dependence on the nonisothermal conditions, but that the size of H_1 remains practically constant and equal to 1.347. Figure 5 compares calculations of the form parameter H made under the same assumptions with formula (2.5). The effect of air dissociation on the size of the form parameter H can be clearly seen.

When there is no dissociation we obtain $H = H_1\psi_h$ from Eq. (2.5).

Thus for the boundary conditions ψ_α = const the form parameter H may be taken as constant over the length of the pipe.

Equations (2.1), (1.15), and (1.10) suffice for calculations of the turbulent boundary layer when there is external flow around bodies, since the velocity at the outer border of the boundary layer W_0 is determined from the conditions of potential flow around bodies and is a known function of the longitudinal coordinate. When dealing with gas flow in the initial section of a pipe the velocity in the core of the stream is the required function, and the equation of continuity

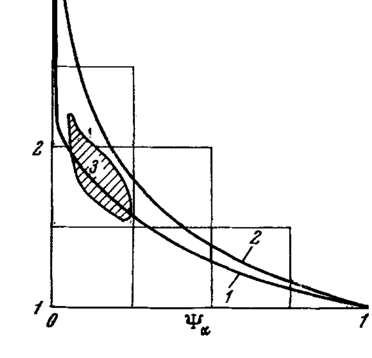

Fig. 3. Comparison of calculations from formula (1.13) with the experiments of N. M. Belyanin; the calculations from formula (1.13) are given by curves 1) for $R^{++} = 300$, 2) for $R^{++} = \infty$, while 3 is the region covered by the experiments of N. M. Belyanin.

must necessarily be employed in order to determine it. The continuity equation for a round pipe may be conveniently written in the form [1]

$$\rho_{01}W_{01} = \rho_0 W_0 (1 - 2\delta^+ / R_0)\,. \qquad (2.6)$$

Introducing the form parameter H, we have

$$R^{++} = {}^1/_4 R_{D1}(W_0 - 1) / H\,. \qquad (2.7)$$

The system of equations (2.1), (1.15), (1.10), and (2.7) is closed and may be solved to give the required relation between the parameters W_0, R^{++}, $c_f/2$ and the length of the pipe x/D.

The law of friction for standard conditions (1.10) may be conveniently approximated by a power function of the form

$$c_{f0} = B / (R^{++})^m\,. \qquad (2.8)$$

Here the values of the coefficients B and m depend on the range of variation of R^{++}.

For R^{++} from 300 to 10^4 we may take B/2 = 0.0128 and m = 0.25.

In this case it is possible to obtain an analytic solution of the initial system of equations for the condition h_w = const. When Eqs. (2.7) and (2.8) are taken into account, Eq. (2.1) may be written in the form

$$\left[\frac{(W_0-1)^m}{W_0} + \frac{(W_0-1)^{m+1}}{W_0^2}(1+H)\right]\frac{dW_0}{dX} = \Psi\frac{B}{2}\frac{(4H)^{1+m}}{(R_{D1})^m} \qquad (2.9)$$

assuming that there is a uniform velocity profile at the entrance to the pipe, allowing for the fact that for the conditions h_w = const, Ψ = const, it follows that H = const (the case of subsonic gas flow is considered),

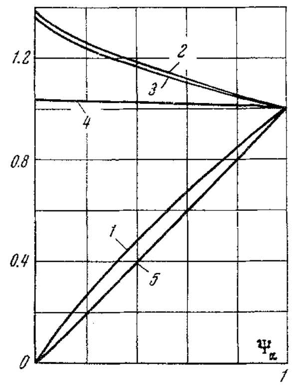

Fig. 4. The effect of nonisothermal conditions on the parameters of an axially symmetric turbulent boundary layer 1) δ^+/δ^+, 2) δ_1^{++}/δ_1^+, 3) δ^{++}, 4) H_1/H_{1is}, 5) δ^+/δ^{++}.

and also assuming that B/2 = 0.0128 and m = 0.25, we may thus obtain from (2.9)

$$\left[(1+H)\frac{5}{4}+1\right]\left[4(W_0-1)^{0.25} - \frac{1}{\sqrt{2}}\ln\frac{(W_0-1)^{0.5}+\sqrt{2}(W_0-1)^{0.25}+1}{(W_0-1)^{0.5}-\sqrt{2}(W_0-1)^{0.25}+1} - \sqrt{2}\,\mathrm{arc\,tg}\frac{\sqrt{2}(W_0-1)^{0.25}}{1-(W_0-1)^{0.5}}\right] - (1+H)\frac{(W_0-1)^{1.25}}{W_0} = \Psi\frac{0.0725H^{1.25}}{R_{D1}{}^{0.25}}X. \qquad (2.10)$$

For the region of subsonic flow of an undissociated gas Eq. (2.10) leads to Eq. (6.35) of paper [1], except that the gas viscosity entering into R_{D1} is determined from the wall temperature.

Figure 6 gives the results of calculations for W_0 as a function of $\xi = X/R_{D1}{}^m$ from Eq. (2.10) for various degrees of dissociation and nonisothermal conditions. Equation (2.10) is valid only for the initial section of the pipe where the dynamic and thermal boundary layers have not joined up.

We shall determine the initial section from the condition that at the end of the stabilized section the thickness of the boundary layer

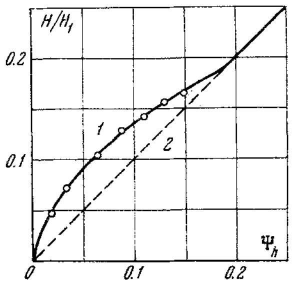

Fig. 5. The effect of air dissociation on the size of the form parameter: 1) calculations from Eq. (2.5), 2) without allowing for dissociation; the points are the calculations made in terms of integral parameters. All the calculations were performed for T_w = 300° K.

becomes equal to the radius of the pipe. Then

$$R_H{}^{++} = R_D\frac{1}{2}\frac{\delta^{++}}{R_0} = R_{D1}\frac{W_{0H}}{2}\frac{\delta^{++}}{R_0}\,, \qquad (2.11)$$

while on the other hand,

$$R_H{}^{++} = \frac{R_{D1}(W_{0H}-1)}{4H}\,. \qquad (2.12)$$

Equations (2.11) and (2.12) give

$$W_{0H} = \frac{1}{1-2\delta^{++}H/R_0}\,. \qquad (2.13)$$

Parameters δ^{++}/R_0 and H may be determined from Fig. 4 in the first approximation. Setting W_{0H} in Eq. (2.10), we may determine the length of the initial region:

$$X_H = \frac{R_{D1}^{0.25}}{\Psi 0.0725H^{1.25}}\left\{\left[(1+H)\frac{5}{4}+1\right]\left[4(W_{0H}-1)^{0.25} - \frac{1}{\sqrt{2}}\ln\frac{(W_{0H}-1)^{0.5}+\sqrt{2}(W_{0H}-1)^{0.25}+1}{(W_{0H}-1)^{0.5}-\sqrt{2}(W_{0H}-1)^{0.25}+1} - \sqrt{2}\times\mathrm{arc\,tg}\frac{\sqrt{2}(W_{0H}-1)^{0.25}}{1-(W_{0H}-1)^{0.5}}\right] - (1+H)\frac{(W_{0H}-1)^{1.25}}{W_0}\right\}. \qquad (2.14)$$

The dashed line in Fig. 6, calculated from Eqs. (2.13) and (2.14), determines the length of the initial section and the region within which

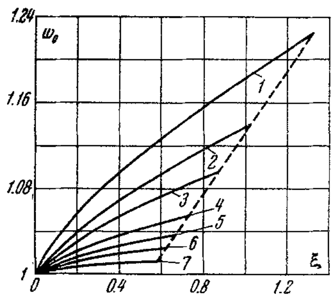

Fig. 6. The dimensionless velocity W_0 in the core of the stream as a function of the pipe length $\xi = XR^{-0.25}$ for various values of ψ_a; the curves 1, ..., 7 correspond to the values ψ_α = 1.06, 0.4, 0.2, 0.137, 0.0875, 0.044. The dashed line determines the boundary of the initial region (calculated from formula (2.14)).

the relations obtained are applicable. We note the marked effect that gas dissociation has on the stabilization length.

The sequence of calculations for a dynamic turbulent boundary layer in the initial cross section of a pipe is as follows.

1. Formulas (1.15) and (2.5) are used to determine the parameters Ψ_∞ and H from the given parameters α_w, α_0, and h_w.

2. The length of the stabilization region X_H is determined from Eqs. (2.13) and (2.14).

3. W_0 as a function of X is calculated from Eq. (2.10) or from the graph in Fig. 6.

4. The value of R^{++} is determined from Eq. (2.7), and the values of the friction coefficients are calculated from Eqs. (2.8) and (1.15) or (1.13).

§3. The development of a warm turbulent boundary layer in the initial section of the pipe. The integral energy equation for the initial section of the pipe may conveniently be written in the form [1]

$$\frac{dR_h^{++}}{dX} + \frac{R_h^{++}}{\Delta h}\frac{d(\Delta h)}{dX} = R_{D1} W_0 \Psi_s S_0. \qquad (3.1)$$

Here S_0 is the Stanton criterion under standard conditions. For the case h_w = const

$$dR_h^{++} / dX = R_{D1} W_0 \Psi_s S_0; \qquad (3.2)$$

on the assumptions which have been made the law of heat transfer has the same form as the law of friction, i.e.,

$$\Psi_s = 4\left[1 + \left(\frac{\psi_h - \Sigma\alpha_{wi}\psi_i}{1 - \Sigma\alpha_{0i}\psi_i}\right)^{1/2}\right]^{-2}. \qquad (3.3)$$

Under standard conditions

$$S_0 = B/2\,(R_h^{++})^m P^n. \qquad (3.4)$$

We now take B/2 = 0.0128, m = 0.25 and n = 0.75. Then the integral of Eq. (3.2) is

$$R_h^{++} = \left[\frac{(1+m)B}{2P^{0.75}} R_{D1}\Psi_s \int_0^X W_0\, dX\right]^{\frac{1}{1+m}}. \qquad (3.5)$$

It follows from Eq. (2.9) that

$$W_0\, dX = \frac{2R_{D1}^m}{B\Psi(4H)^{1+m}} \times$$

$$\times \left[(W_0 - 1)^m + \frac{(W_0 - 1)^{m+1}}{W_0}(1 + H)\right] dW_0. \qquad (3.6)$$

Consequently

$$R_h^{++} = \frac{R_{D1}(W_0 - 1)}{4HP^{0.8n}}\left\{2 + H - \right.$$

$$- \frac{(1+H)\,1.25}{(W_0 - 1)^{1.25}}\left[4(W_0 - 1)^{0.25} - \right.$$

$$- \sqrt{2}\,\mathrm{arc\,tg}\,\frac{\sqrt{2}(W_0 - 1)^{0.25}}{1 - (W_0 - 1)^{0.5}} -$$

$$\left. \left. - \frac{1}{\sqrt{2}} \ln \frac{(W_0 - 1)^{0.5} + \sqrt{2}(W_0 - 1)^{0.25} + 1}{(W_0 - 1)^{0.5} - \sqrt{2}(W_0 - 1)^{0.25} + 1}\right]\right\}^{0.8}. \qquad (3.7)$$

Knowing W_0 as a function of X, as determined by Eq. (2.10), we may find the local values of the criterion R_h^{++} from Eq. (3.7), while the local value of the Stanton criterion S is determined from formulas (3.4) and (3.3).

Equations (2.7), (2.8), (3.4), and (3.7) may be used to obtain the relation between the local values of the friction and heat transfer coefficients for the conditions under consideration:

$$\frac{c_f}{2S} = P^{0.75}\left\{2 + H - \frac{(1+H)\,1.25}{(W_0 - 1)^{1.25}}\left[4(W_0 - 1)^{0.25} - \right.\right.$$

$$- \sqrt{2}\,\mathrm{arc\,tg}\,\frac{\sqrt{2}(W_0 - 1)^{0.25}}{1 - (W_0 - 1)^{0.5}} -$$

$$\left.\left. - \frac{1}{\sqrt{2}} \ln \frac{(W_0 - 1)^{0.5} + \sqrt{2}(W_0 - 1)^{0.25} + 1}{(W_0 - 1)^{0.5} - \sqrt{2}(W_0 - 1)^{0.25} + 1}\right]\right\}^{0.2}. \qquad (3.8)$$

Calculations made on the basis of Eq. (3.8) show that the pressure gradient in the initial section exerts no marked effect on the Reynolds similarity.

The proposed method of calculation may be extended to the case where the distribution h_w along the channel is arbitrary, by subdividing the channel into separate sections with constant gas enthalpy at the wall.

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